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ON THE STABILITY OF SPACE-PERIODIC CONVECTIVE FLOWS IN A VERTICAL LAYER WITH SINUOUS BOUNDARIES

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The problem of convection in a vertical layer with harmonically distorted boundaries is examined by perturbation theory methods for a small amplitude of sinuosity. The solutions obtained are applicable both in the stability region as well as in the supercritical region of the plane-parallel flow. The stability of the solutions found is investigated with respect to a certain class of space-bounded perturbations that are not necessarily space-periodic. The method of amplitude functions [1], generalized to the case of curved boundaries, is used. The Grashof critical number is found as a function of the period of sinuosity and the form of the neutral curve for the space-periodic motions and their stability region are obtained. It is established that if the deformation period of the boundaries is close to the wavelength of the critical perturbation for the plane-parallel flow or is twice as great, then as the Grashof number grows stability loss does not occur and the motion's amplitude changes continuously (cf. [2 — 4]). A comparison is made with the results of the numerical calculation in [5]. An attempt was made in [6] to construct a stationary periodic motion in a layer with weakly-deformed boundaries, in the form of series in powers of a small sinuosity amplitude. However, the solution obtained diverges in a neighborhood of the neutral curve of the plane-parallel flow and approximates unstable motion in the supercritical region of the unperturbed problem. Flows under a finite sinuosity amplitude are calculated by the net method in [5] wherein the stability of the flows was investigated as well, but only with respect to perturbations with wave numbers that are multiples of $2\pi/l$, where l is the length of the calculated region.

1. We examine the planar motion of a liquid in an infinite vertical layer on whose rigid boundaries

$$x = \pm d\,(1 - \eta \cos k_0 y \,/\, d)$$

different constant temperatures $T = \mp \Theta$ are maintained. We write the system of convection equations in dimensionless variables as

$$\left(L - \frac{\partial}{\partial t} M\right) U + \frac{1}{2} D(U, U) = 0 \tag{1.1}$$

$$U = \begin{Vmatrix} \psi \\ T \end{Vmatrix}, \quad L = \begin{Vmatrix} \Delta^2 & -G\dfrac{\partial}{\partial x} \\ 0 & P^{-1}\Delta \end{Vmatrix}, \quad M = \begin{Vmatrix} \Delta & 0 \\ 0 & 1 \end{Vmatrix}$$

$$D(U_1, U_2) = \left\| \begin{array}{c} \dfrac{D(\psi_1, \Delta\psi_2)}{D(x,y)} + \dfrac{D(\psi_2, \Delta\psi_1)}{D(x,y)} \\ \dfrac{D(\psi_1, T_2)}{D(x,y)} + \dfrac{D(\psi_2, T_1)}{D(x,y)} \end{array} \right\|$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \frac{D(f,g)}{D(x,y)} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

Here ψ is the stream function, T is the temperature, P is the Prandtl number, and G is the Grashof number determined from the mean halfwidth d of the layer and half the temperature difference Θ. The boundary conditions are

$$x = \pm(1 - \eta\cos k_0 y), \quad T = \mp 1, \ \psi = \partial\psi / \partial x = 0 \tag{1.2}$$

$$y \to \pm\infty, \quad |\psi| < \infty, \quad |T| < \infty \tag{1.3}$$

In what follows the parameter η is assumed small.

2. At first we construct the stationary solutions of system (1.1) –(1.3), satisfying the periodicity condition

$$U(x, y + 2\pi / k_0) = U(x, y) \tag{2.1}$$

We introduce a coordinate transformation rectifying the layer's curvilinear boundaries

$$y' = y, \quad x' = x / (1 - \eta\cos k_0 y) \tag{2.2}$$

In the new variables the boundary conditions are specified on the flat boundaries

$$x' = \pm 1, \ \psi = \partial\psi / \partial x' = 0, \ T = \mp 1 \tag{2.3}$$

while the convection Eqs. (1.1) become inhomogeneous and acquire the form

$$LU + \frac{1}{2}D(U,U) = \sum_{n=1}^{\infty} \eta^n \sum_{m=-n}^{n} (A_{mn}U + \tag{2.4}$$

$$\frac{1}{2}B_{mn}(U,U))\, e^{imk_0 y'}$$

$$A_{-m,n} = \bar{A}_{mn}, \quad B_{-m,n} = \bar{B}_{mn}$$

(henceforth the primes will be omitted for simplicity). The indices m and n are of like parity.

When $\eta = 0$ problem (2.3) and (2.4) possesses, for any G, a solution corresponding to plane-parallel motion

$$\psi_0 = -\frac{G}{24}(1 - x^2)^2, \quad T_0 = -x \tag{2.5}$$

For moderate values of the Prandtl number (being precisely the case to be analyzed subsequently) the solution given becomes unstable with respect to monotonically growing perturbations of period $2\pi / k$ when exceeding the threshold value of the Grashof number $G_0(k)$; the neutral curve $G_0(k)$ has a minimum $G = G_c$ for some $k = k_c$. Stationary solutions, periodic in y, exist in the region $G > G_c$, in addition to solution (2.5) [7].

For small $\eta \neq 0$ we seek the solution of problem (2.3) and (2.4) as a power series in η, choosing solution (2.5) as the zero approximation

$$U = \sum_{n=0}^{\infty} \eta^n U^{(n)}, \quad U^{(0)} = (\psi_0, T_0) \tag{2.6}$$

Substituting expansion (2.6) into (2.3) and (2.4), in each order with respect to η we obtain an inhomogeneous boundary problem for the determination of $U^{(n)}$

$$LU^{(n)} + D(U^{(0)}, U^{(n)}) = -\frac{1}{2}\sum_{p=1}^{n-1} D(U^{(p)}, U^{(n-p)}) + \tag{2.7}$$

$$\sum_{l=1}^{n}\sum_{m=-l}^{l}\left(A_{ml}U^{(n-l)} + \frac{1}{2}\sum_{p=0}^{n-l} B_{ml}(U^{(p)}, U^{(n-l-p)})\right) \times e^{imk_0 y}$$

$$x = \pm 1, \quad \psi^{(n)} = \partial\psi^{(n)} / \partial x = T^{(n)} = 0 \tag{2.8}$$

$$y \to \pm\infty, \quad |\psi^{(n)}| < \infty, \quad |T^{(n)}| < \infty$$

The indices m and n in (2.7) are of like parity. The boundary-value problem (2.7) and (2.8) is solvable for any right hand side if

$$G \neq G_0(nk_0), \quad n \geqslant 1 \tag{2.9}$$

According to the linear stability theory of a plane-parallel flow, other branches of the neutral curve, lying considerable higher with respect to the Grashof number and connected with thermal waves [8] and the Tollmien — Schlichting waves [9], correspond to running perturbations with nonzero frequency. For this reason their existence does not impose additional constraints on the solvability of problem (2.7).

In general, problem (2.7) and (2.8) has no solutions on curve $G = G_0(nk_0)$, and as G approaches $G_0(nk_0)$ the function $|U^{(n)}| \to \infty$. This circumstance, established numerically in [6], is due to the fact that when $G = G_0(nk_0)$ the boundary-value problem (2.7) and (2.8) has a nontrivial solution when the right hand side of the eqution is zero. The divergence of function $U^{(n)}$ attests to the ill-posedness of expansion (2.6) in powers of η close to the neutral curve, where the distortion in the motion's plane-parallelism takes place not only because of the sinuosity of the boundaries but also as a result of the crisis in flow (2.5). Thus, in the region $G \gtrsim G_0$ the amplitude of the nonplane-parallel component of the motion is $\varepsilon \gg \eta$, and as $\eta \to 0$ the solution desired must pass not into a plane-parallel one but into a secondary flow in a layer with flat boundaries. However, if the noncriticality of $G - G_0(nk_0)$ and the magnitude of η are small, then amplitude ε must be small. In this case the solution can also be constructed with the aid of expansions with respect to a small parameter as which we should choose not η but ε.

We restrict the analysis to the cases $G \approx G_0(k_0)$ and $G \approx G_0(2k_0)$. Let G lie in a neighborhood of $G_0(k_0)$. Henceforth we assume $G - G_0 = O(\varepsilon^2)$ and we introduce the notation

$$G - G_0 = \varepsilon^2 G^{(2)} \tag{2.10}$$

We seek a stationary periodic solution of problem (2.3) and (2.4) in the form

$$U - U^{(0)} = \sum_{n=1}^{\infty} \varepsilon^n U^{(n)}, \quad U^{(0)} = (\psi_0, T_0) \tag{2.11}$$

We represent the connection between the quantities ε, η and G as

$$\eta = \sum_{n=1}^{\infty} \varepsilon^n \eta^{(n)}(G) \quad (2.12)$$

Substituting (2.10) – (2.12) into (2.4) and (2.3), in the n-th order of ε we obtain a boundary-value problem from whose solvability condition (orthogonality of the equation's right hand side to the solution of the adjoint problem) we find the coefficient $\eta^{(n)}$. When $n = 1$ we obtain an equation whose solvability condition yields $\eta^{(1)} = 0$, and the solution has the form

$$U^{(1)} = a u_1^{(1)}(x) e^{ik_0 y} + \bar{a} u_{-1}^{(1)}(x) e^{-ik_0 y}$$

The subscript corresponds to the number of the harmonic $e^{imk_0 y}$. The complex coefficient a whose magnitude depends upon the norming of function $u_1^{(1)}$ has to be determined in the subsequent orders.

Analogously, from the solvability condition for the equations in the second order we obtain $\eta^{(2)} = 0$, while in the third order the connection between $\eta^{(3)}$ and a is

$$JG^{(2)}a - S\,|a|^2 a + D\eta^{(3)} = 0 \quad (2.13)$$

Because they are cumbersome we do not derive the explicit expressions for the coefficients J, S and D which are scalar products of the inhomogeneous right hand sides of the equations and the solution of the adjoint linear problem. To find the functions $u_1^{(1)}$, $u_0^{(2)}$ and $u_2^{(2)}$ occurring in these expressions we used the Runge – Kutta method. Let us present the numerical values of the coefficients for $P = 1$, $k_0 = k_c = 1.38$, and norming $\mathrm{Re}\,\vartheta_{-1}^{(1)\prime} = -1$, where $u_1^{(1)} = (\varphi_1^{(1)}, \vartheta_1^{(1)})$: $J = 0.059$, $S = 2797$, $D = 18.8$.

Multiplying (2.13) by ε^3 and denoting $a_1 = \varepsilon a$, we obtain an approximate equation connecting the amplitude of the nonplane-parallel flow component a_1 with the quantity $G - G_0$ and the boundary sinuosity parameter η

$$J(G - G_0)\,a_1 - S\,|a_1|^2 a_1 + D\eta = 0 \quad (2.14)$$

Introducing the notation

$$\gamma_0 = G - G_0, \quad Z = a_1\sqrt{S/J}, \quad \eta_1 = \eta D\sqrt{S/J^3} \quad (2.15)$$

we rewrite (2.14) as

$$\gamma_0 Z - |Z|^2 Z + \eta_1 = 0 \quad (2.16)$$

This equation is of a very general nature. It describes the variation of the critical mode amplitude close to the stability threshold in the presence of some stationary external force (in the present case, the boundary distortions). An analogous equation was obtained previously in [2 –4]. In the absence of boundary distortions $(\eta_1 = 0)$ Eq. (2.16) has the solution $Z = 0$ corresponding to a plane-parallel flow, while when $\gamma_0 > 0$ it has as well a set of solutions $Z = \sqrt{\gamma_0}\,e^{i\varphi}$ corresponding to the secondary flow. The phase φ determining the vortex center positions relative to the layer walls is arbitrary because of the problem's translational symmetry.

Boundary distortions $(\eta_1 \neq 0)$ lead to the removal of phase degeneracy: only real solutions of the cubic Eq. (2.16) are left. Fig. 1 presents the graphs of $Z(\gamma_0)$

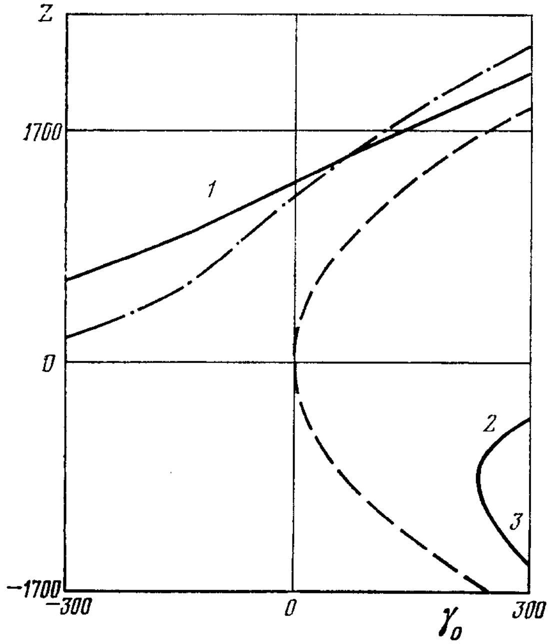

Fig. 1

when $\eta = 0.02$. For the motions corresponding to branch *1* the vortex centers are located in the wide sections of the layer, while for branches *2* and *3*, in the narrow sections. When $\gamma_0 = 0$ the quantity $Z = \eta_1^{1/3}$; the coordinates of the point of confluence of branches *2* and *3* are: $\gamma_0 = 3(\eta_1 / 2)^{2/3}$, $Z = -(\eta_1 / 2)^{1/3}$. For large $|\gamma_0|$ the solutions of (2.16) differ from the corresponding solutions for $\eta_1 = 0$ (shown in Fig. 1 by the dashed line) by an amount $O(\eta_1)$. The dash-dotted curve represents the solutions obtained by the net method in [5] for the same problem parameter values corresponding to branch *1* (the remaining branches, as is shown in Sect. 4, are unstable). The difference between the results of the analytical and numerical calculations with $G = G_0$ is 7%.

Let us now consider the stationary solutions of problem (2.3)–(2.4) with G close to $G_0(2k_0)$. In this case the function $U^{(1)}$ in expansion (2.6) remains finite, i.e., the intensity of the first harmonic is of the order of η. However, function $U^{(2)}$ diverges since the intensity of the second harmonic when passing through the critical (for the plane-parallel flow) Grashof number becomes much greater than η^2. We set

$$G - G_0(2k_0) = \varepsilon^2 G^{(2)} \tag{2.17}$$

and we introduce the expansion

$$U - U^{(0)} = \sum_{n=1}^{\infty} \varepsilon^{n/2} U^{(n)}, \quad U^{(0)} = (\psi_0, T_0) \tag{2.18}$$

$$\eta = \sum_{n=1}^{\infty} \varepsilon^{n/2} \eta^{(n)} \tag{2.19}$$

As before, we substitute (2.17)–(2.19) into (2.4) and (2.3) and solve the equations resulting in each order of $\varepsilon^{1/2}$. The coefficients $\eta^{(n)}$ are determined from the solvability conditions for the corresponding equations. It turns out that in expansion (2.18) the first nonzero term for the second harmonic has $n = 2$, while for the first harmonic, $n = 3$. In expansion (2.19) the first nonzero coefficient is $\eta^{(3)}$. Omitting the details of the derivation, we present the equation for the amplitude a_2 of the second harmonic

$$J(G - G_0(2k_0))\, a_2 - S\, |a_2|^2 a_2 + D_1 \eta^2 = 0$$

where J and S are the same coefficients as in Eq. (2.13). The numerical value of the new coefficients is $D_1 = -189.7$ for $P = 1$ and $k_0 = k_c / 2$. Thus, when passing through the threshold number $G_0(2k_0)$ a rapid growth takes place in the intensity of the second harmonic, which now for $G = G_0(2k_0)$ becomes of the order of $\eta^{2/3}$ and exceeds in magnitude the first harmonic which is of the order of η. One of the vortices of the second harmonic is located in the layer's wide region and strengthens the fundamental mode, while the second is located in the narrow region. The flow pattern described qualitatively coincides with the one obtained by a numerical solution with the net method of [5].

3. We go on to investigate the stability of the space-periodic motions constructed in Sect. 2. For normal perturbations $Ve^{-\lambda t}$ imposed on the stationary solution U we obtain the eigenvalue problem

$$(L + \lambda M)V + D(U, V) = \sum_{n=1}^{\infty} \eta^n \sum_{m=-n}^{n} (A_{mn}V + \tag{3.1}$$

$$B_{mn}(U, V) + \lambda C_{mn}V)\, e^{imk_0 y}, \quad V = \begin{Vmatrix} \Phi \\ \Theta \end{Vmatrix}$$

$$x = \pm 1, \quad \Phi = \partial\Phi / \partial x = \Theta = 0$$

$$y \to \pm\infty, \quad |\Phi| < \infty, \quad |\Theta| < \infty$$

We examine a class of solutions of problem (3.1), representable as

$$V(x, y) = W(x, y)\, e^{iky}, \quad W(x, y + 2\pi / k_0) = W(x, y) \tag{3.2}$$

analogously to the case of ordinary differential equations with periodic coefficients [10]. Functions (3.2) represent a discrete group of translations along the y-axis by distances multiples of $2\pi / k_0$ in accord with the symmetries of problem (3.1). The real parameter k is determined to within an integral multiple of k_0: $k = K + nk_0$, $|K| \leqslant k_0 / 2$.

When $\eta = 0$ the solutions of system (3.1)

$$V_0 = w_{00}(x)\, e^{iky} \tag{3.3}$$

describe the perturbations in the absence of boundary sinuosity. The function $\lambda_0(k)$ $(\lambda_0 \equiv \lambda|_{\eta=0})$ is positive when $G < G_c$, vanishes at point $k = k_c$ when $G = G_c$, and is negative inside a certain interval Δk in the neighborhood of k_c when $G > G_c$. When $\eta \neq 0$ we seek the solution of problem (3.1) in the form

$$V = \sum_{n=0}^{\infty} V_n \eta^n, \quad \lambda = \sum_{n=0}^{\infty} \lambda_n \eta^n \tag{3.4}$$

choosing function (3.3) and the decrement $\lambda_0(k)$ as the zero approximation. Then the Fourier expansion of function V_n is

$$V_n(x, y) = e^{iky} \sum_{m=-n}^{n} w_{mn}(x)\, e^{imk_0 y}$$

If the inequality

$$\lambda_0(k) \neq \lambda_0(k + mk_0) \tag{3.5}$$

is fulfilled for any $m = \pm 1, \pm 2, \ldots$, the coefficient λ_n is determined from the solvability condition of the equation for $w_{n0}(x)$. It can be shown that the coefficients $\lambda_n = 0$ for odd n. We can convince ourselves as well that the shift in the critical wave number resulting from boundary distortion is of the order of η^2. The shift in the critical Grashof number is determined by the quantity $\lambda_2(k_c, G_c)$ and equals

$$G - G_c = \frac{\lambda_2(k_c, G_c)}{(d\lambda / dG)_{k_c, G_c}} \eta^2$$

Flow stabilization and destabilization correspond to the values $\lambda_2 > 0$ and $\lambda_2 < 0$, respectively; when $P = 1$ the derivative $(d\lambda / dG)_{k_c, G_c} = -0.0235$. The dependence of the quantity $\lambda_2(k_c, G_c)$ on k_0 is shown in Fig. 2 ($P = 1$).

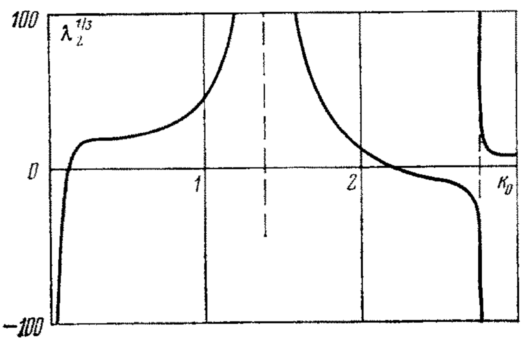

Fig. 2

4. The cases $k_0 = 0$, k_c and $2k_c$, for which the magnitude of λ_2 becomes infinite, require a special analysis. It can be shown that the coefficients λ_{2n} diverge in the subsequent orders when $k_0 = k_c / n$ and $k_0 = 2k_c / n$. The divergence of λ_{2n} is due to the following reasons. The equality $\lambda_0(k_c) = \lambda_0(k_c - nk_0)$ holds at the points $k_0 = 2k_c / n$, which leads to a violation of condition (3.5) for $k = k_c$. When considering these cases it is necessary to make some modifications in perturbation theory. The growth of the expansion coefficients in (3.4) as $k_0 \to 0$ also is caused by the proximity of the quantities $\lambda_0(k)$ and $\lambda_0(k + mk_0)$. The equality $G_0(nk_0) = G_c$ is achieved at points $k_0 = k_c / n$, as a result of which condition (2.9) is violated and expansion (2.6) is inapplicable for the main flow.

At first we consider those particular cases for which condition (2.9) is fulfilled ($k_0 \to 0$ and $k_0 = 2k_c / n$), where n is odd. As $k_0 \to 0$ (i.e., when the layer's thickness changes very slowly) the perturbations localized in the layer's wide part will possess damping decrements close to the decrements of the perturbations of a plane-parallel flow in a plane channel of halfwidth $1 + \eta$. Instability appears when

$$G = G_c / (1 + \eta)^3$$

i.e., $G - G_c \approx -3\eta G_c$.

The equality $\lambda_0(k_c) = \lambda_0(k_c - k_0)$ holds when $k_0 = 2k_c$. In this case we should choose as the zero approximation the superposition

$$V_0 = c_1 w_{0,0}(x) e^{ik_c y} + c_2 w_{0,-1}(x) e^{-ik_c y}, \quad w_{0,-1} = \overline{w}_{0,0}$$

with undetermined coefficients (see [11], for instance). In the next order the solvability conditions for $w_{1,0}$ and $w_{-1,1}$ yield the system of equations

$$I\lambda_1 c_1 - Fc_2 = 0, \quad Fc_1 - I\lambda_1 c_2 = 0 \qquad (4.1)$$

We do not cite the explicit expressions for coefficients F and I; when $P = 1$ and $k_c = 1.38$ we have $F = -45.5$ and $I = 2.56$. System (4.1) has two solutions $c_1 = \pm c_2$ for which $\Phi_0(0, y)$ is, respectively, an even and an odd function of y; moreover

$$\lambda_1 = \pm F / I$$

We see that destabilization obtains for an even mode, therefore, it is stimulated for smaller values of G than in an odd mode, which is consistent with the results of the numerical calculations in [5]. It can be shown analogously that when $k_0 = 2k_c / n$ (n is odd) nonzero λ_n appear for even and odd modes.

We now consider the case when $k_0 \approx k_c$. Expansion (2.6) is inapplicable for values of G close to G_c, which are of the greatest interest from the viewpoint of stability, and the amplitude of the nonplane-parallel component of the stationary motion is described by Eq. (2.14). In order to investigate the motion's stability with respect to perturbations of the same periodicity as the flow itself, it is sufficient to repeat the derivation of Eq. (2.14), taking amplitude a as a function of the slow time $\varepsilon^2 t$ (see [1]). We obtain an equation that differs from (2.14) by a term $I\partial a_1 / \partial t$ in the right hand side (I is the same coefficient as in Eqs. (4.1)). Using notation (2.15), we reduce the amplitude equation to the form

$$\partial Z / \partial T = \gamma_0 Z - |Z|^2 Z + \eta_1 \quad (T = Jt / I)$$

For perturbations $ze^{-\lambda T}$ imposed on the stationary solution Z_0 satisfying (2.16) we obtain the equation

$$\lambda z + \gamma_0 z - 2Z_0^2 z - Z_0^2 \bar{z} = 0$$

It has two solutions

$$z = -\bar{z}, \ \lambda = Z_0^2 - \gamma_0; \quad z = \bar{z}, \quad \lambda = 3Z_0^2 - \gamma_0$$

The first type of perturbations, the most dangerous, corresponds to a shift in the phase of function Z (i.e., to a spatial displacement of the whole vortex system with unchanged intensity); the second type corresponds to a change in its amplitude. We can convince ourselves that only branch *1* is stable; branch *3* is unstable with respect to perturbations of the first type, while branch *2*, to perturbations of both types. We take note of the fact that the first type of perturbations is connected with the complex-valuedness of function Z and is absent in the problems in [2, 4] for which branch *3* is stable. The conclusion on the stability of branch *1* and on the instability of the remaining branches is confirmed by the numerical calculations in [5].

We go on to investigate the stability of the stationary motions corresponding to branch *1* with respect to perturbations of the general form (3.2). We note that the net method permits us to analyze only the perturbations with wave numbers k that are multiples of $2\pi / l$, where l is the length of the calculated region. The situation when the parameters k and $2\pi / l$ are arbitrarily related, in particular, are close, cannot be handled by this method.

Since in region $G \approx G_c$ only perturbations with wave numbers close to k_c (i.e., with small K) are of interest from the viewpoint of stability, to investigate the stability we apply the many-scale method in the form suggested in [1]. Omitting the details of derivation, we present the final form of the equation for the amplitude function a_1 depending on the slow time variable $\varepsilon^2 t$ and the slow coordinate variable εy

$$I\frac{\partial a_1}{\partial t}=R\frac{\partial^2 a_1}{\partial y^2}+J(G-G_c)a_1-S|a_1|^2a_1+D\eta e^{i(k_0-k_c)y}$$

When $P=1$ the new coefficient's value is $R=14.25$. Using notation (2.15), we obtain

$$\partial Z/\partial T=\partial^2 Z/\partial Y^2+\gamma Z-|Z|^2Z+\eta_1 e^{iK_0 y}$$
$$Y=y\sqrt{J/R},\quad K_0=k_0-k_c,\quad \gamma=G-G_c,\quad T=Jt/I$$

The amplitude function

$$Z_0=re^{iK_0 y},\quad r^3-(\gamma-K_0^2)r-\eta_1=0 \tag{4.2}$$

corresponds to the stationary solutions with period $2\pi/k_0$.

For small perturbations $ze^{-\lambda T}$ imposed on solution (4.2) we obtain the equation

$$\lambda z+d^2z/dY^2+\gamma z-2Z_0^2z-Z_0^2\bar{z}=0$$

whose solutions have the form

$$z=ae^{i(K_0+K)y}+be^{i(K_0-K)y}$$

while the two branches of the decrement are described by the formula

$$\lambda_\pm=K^2+r^2+\eta_1/r\pm\sqrt{4K^2K_0^2+r^4}$$

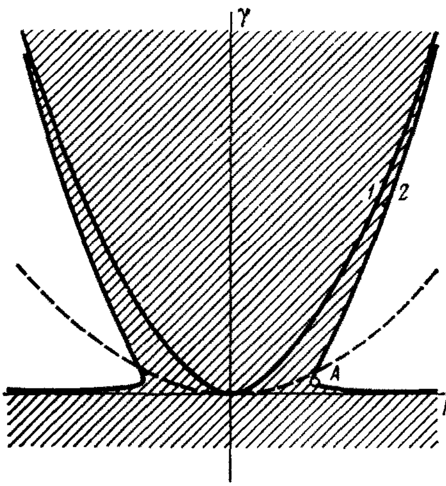

Fig. 3

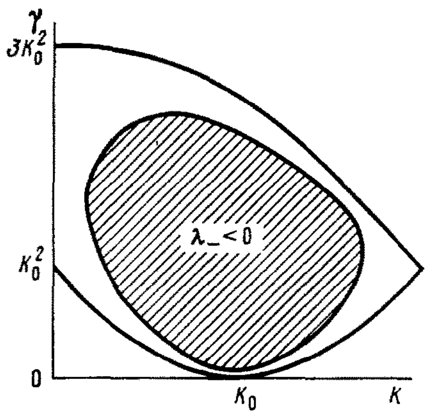

Fig. 4

The stability region for the stationary periodic motions when k_0 is close to k_c is shaded in Fig. 3; its boundary (line *2*) is described by the equation

$$\gamma-2r^2+r^4/(4K_0^2)=0$$

As $\eta\to 0$ the stability boundary splits up into two parts: $\gamma=0$ for the plane-parallel flow and $\gamma=3K_0^2$ for the secondary flow in accordance with [12] (curve *1*). The coordinates of point A are: $K_A^3=2.47\eta_1$, $\gamma_A=0.76\,K_A^2$. A loss of stability does not take place in the interval $|K_0|<K_A$ when passing through the critical (for the plane-parallel flow) Grashof number G_c, and the motion's amplitude changes continuously. When $|K_0|>K_A$, as the Grashof number increases the periodic motion at first loses stability and then becomes stable again. The neutral curve of the plane-parallel flow is shown as a dashed line.

Fig. 4 shows the neutral curve (the boundary is $\lambda_- = 0$) for the flow with $k_0 = k_c + K_0$ (the instability region has been shaded). As $\eta \to 0$ the curve splits into two parts: $\gamma = (K - |K_0|)^2$ for $0 < \gamma < K_0^2$ (the neutral curve of the plane-parallel flow) and $\gamma = 3K_0^2 - K^2/2, K = 0$ when $\gamma > K_0^2$ (the neutral curve of the secondary flow). The interval of unstable wave numbers is maximal when $\gamma = K_0^2$ and vanishes when $\gamma = 3K_0^2$. In the region $\gamma > 3K_0^2$ the secondary motion remains stable with respect to any planar perturbations.

Boundary distortion ($\eta_1 \neq 0$) narrows down the instability region of the space-periodic motions in accordance with Fig. 3. The analysis of stability for the case $k_0 \approx k_c/2$ is analogous to the one above for the case $k_0 \approx k_c$; only the connection between η_1 and η is changed: now $\eta_1 = O(\eta^2)$.

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